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Numerical Algorithms for Axisymmetric Fokker–Planck–Landau Operators

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We establish a simplified form of the axisymmetric Fokker–Planck–Landau operator. In doing so, we derive a weak formulation of this operator on which we get directly the conservation properties and the decay of the entropy as it is the case for the three-dimensional operator. A symmetrized version of this formulation leads to a class of numerical schemes which satisfy these physical properties at the discrete level. Fast numerical algorithms used in previous works are shown to be efficient in the cylindrical geometry. Finally, some numerical tests are presented at the end of this paper. © 2000 Academic Press

1. INTRODUCTION

The Fokker–Planck–Landau (FPL) equation is used for the description of binary collisions between charged particles, for which the interaction potential is the long-range Coulomb interaction. If $f(t, v)$ is the distribution function of particles (assumed to be spatially homogeneous), then the homogeneous FPL equation is

$$\frac{\partial f}{\partial t} = Q(f, f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} \Phi(v - v')(f(v')\nabla f(v) - f(v)\nabla f(v'))\, dv' \right). \quad (1.1)$$

$Q(f, f)$ is the FPL collision operator. $\Phi(v)$ is the 3×3 matrix

$$\Phi(v) = K(|v|)|v|^2 S(v), \qquad S(v) = \mathrm{Id}_3 - \frac{v \otimes v}{|v|^2}. \quad (1.2)$$

$S(v)$ is the orthogonal projector onto the plane orthogonal to v and $K(|v|)$ is an arbitrary positive function which usually takes the value $|v|^\gamma$. γ is a real parameter which leads to the usual classification in hard potentials ($\gamma > 0$), Maxwellian molecules ($\gamma = 0$), or soft potentials ($\gamma < 0$). This latter case involves the Coulombian case itself (i.e., $\gamma = -3$). For a precise physical context of FPL operators, see [17, 13, 21, 22].

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The FPL collision operator is the limit of the Boltzmann operator for a sequence of scattering cross section which converges in a convenient sense to a delta function at zero scattering-angle [1, 9]. In the case of a Coulomb interaction, the FPL collision operator is obtained as the leading term of the cut-off Boltzmann operator when the parameter of the cut-off tends to zero [7]. The problem of the convergence (in some sense) of the solutions to the homogeneous Boltzmann equation towards those of the FPL equations in the grazing collisions limit has been investigated in a recent work [24]. This last study includes the physical interesting case of Coulomb interactions. In this paper we are concerned with numerical aspects of the FPL operator in the particular case when the distribution function has a cylindrical symmetry with respect to the velocity variable. This geometry is a natural context for the laser-produced plasma interactions.

The FPL operator can be written in a weak formulation as

$$\begin{aligned}\int_{\mathbb{R}^3} Q(f,\, f)(v)\psi(v)\,dv = -\frac{1}{2}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,f(v')(\nabla\psi(v)\\ -\nabla\psi(v'))^T\Phi(v-v')(\nabla(\ln f)(v)\\ -\nabla(\ln f)(v'))\,dv\,dv'\end{aligned} \tag{1.3}$$

for any smooth test function ψ. From this duality relation, one can derive (at least formally) the following physical properties on $Q(f,\, f)$:

(i) Conservation of mass, momentum, and energy,

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3} f(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}dv\right) = \int_{\mathbb{R}^3} Q(f,\, f)(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}dv = 0. \tag{1.4}$$

(ii) The equilibrium functions (or steady states), that is, the positive functions f such that $Q(f,\, f) = 0$ are Maxwellians,

$$M_{\mathcal{N},u,T}(v) = \frac{\mathcal{N}}{\left(2\pi v_{th}^2\right)^{3/2}}\exp\left(\frac{-|v-u|^2}{2v_{th}^2}\right). \tag{1.5}$$

$\mathcal{N}$ is the density number of particles and v_{th} is the thermal velocity which is linked to the temperature T of the gas by the relation

$$v_{th} = \sqrt{\frac{kT}{m}},$$

where k is the Boltzmann constant, m the mass of one particle, and u is the mean velocity of the particles.

We shall say that a function ψ is a collisional invariant if

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3} f(t,v)\psi(v)\,dv\right) = \int Q(f,\, f)(v)\psi(v)\,dv = 0 \qquad \forall f > 0. \tag{1.6}$$

Then from (1.3) and using the nullsapce of the matrix $\Phi(v-v')$, we deduce that the collisional invariants are the functions $\psi(v)$ such that $\nabla\psi(v) - \nabla\psi(v')$ is colinear to $v - v'$ for all v and v' in $\mathbb{R}^3$. Such functions are linear combinations of mass, momentum, and energy,

i.e., 1, v, and $|v|^2$. Again from (1.3), we now observe that $Q(f, f) = 0$ if and only if $\ln f$ is a collisional invariant. The property (ii) is then equivalent to saying that the only collisional invariants are linear combinations of 1, v, and $|v|^2$. We refer to [6] for details.

(iii) H-theorem,

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3} f(t, v) \ln f(t, v)\, dv \right) = \int_{\mathbb{R}^3} Q(f, f)(v) \ln(f(v))\, dv \leq 0, \tag{1.7}$$

translating the decay of the kinetic entropy $\mathcal{H} = \int_{\mathbb{R}^3} f \ln f\, dv$.

These are three fundamental properties that govern the evolution of particles in the plasma. A good numerical discretization of the FPL operator has to obey this physical behavior, and so has to satisfy discrete analogues of the three properties. Such a discretization ensures for instance the relaxation (in time) of the distribution function to the right Maxwellian. Various works have been concerned with numerical schemes for the FPL equation. We mention in particular [3] for the isotropic case and [19, 20, 25] for the axisymmetric problems. We also refer to [10] for a time-implicit scheme to solve the FPL equation and to [14] for a mass conserving finite volume scheme. All these works satify at least one (usually the conservation of mass) of the above physical properties but never simultaneously both of them. Morever, such schemes are of quadratic complexity and the computational cost is big. In the recent past, discretizations of the three-dimensional FPL operator that satisfy all the above physical properties have been performed first in [6], and second in [5, 15] using fast numerical algorithms. The use of a rapid numerical scheme is of crucial importance when one wants to solve realistic problems in plasma physics, i.e., with the presence of the transport term or/and a self-consistent force term (Vlasov–Poisson–FPL system). In [23] a difference scheme to solve the Vlasov–Fokker–Planck system is introduced and a convergence result of this scheme is established in one dimension (for space and velocity). The operator considered in [23] is a linear and one-dimensional collisional operator. Finally numerical analyis and approximations of other Fokker–Planck models in plasma physics have been investigated in [8].

In this paper we are concerned with the expressions of the collision operator and its numerical discretizations when the distribution function f presents a cylindrical symmetry. More precisely, let (I, J, K) be a canonical basis of $\mathbb{R}^3$, and let us write any vector $v \in \mathbb{R}^3$ with cylindrical coordinates in the following way

$$v = v_\parallel I + v_\perp U, \qquad U = (\cos\theta) J + (\sin\theta) K,\ v_\parallel \in \mathbb{R},\ v_\perp \in \mathbb{R}_+,\ \theta \in [0, 2\pi]. \tag{1.8}$$

A function f of $v \in \mathbb{R}^3$ presents a cylindrical symmetry (or is axisymmetric) if and only if f depends only on the two components $v_\parallel \in \mathbb{R}$ and $v_\perp \in \mathbb{R}_+$. In all the sequel the distribution functions are supposed to be axisymmetric. We denote by $\partial_\parallel \psi$ and $\partial_\perp \psi$ the partial derivatives of ψ with respect to $v_\parallel$ and $v_\perp$, when ψ is axisymmetric. Finally, if we set $V = (v_\parallel, v_\perp)$ and $V' = (v'_\parallel, v'_\perp)$, then we prove that there is still a weak formulation similar to (1.3) according to (2.9).

Replacing ψ successively by 1, $v_\parallel$, $v_\parallel^2 + v_\perp^2$ in the weak formulation (2.9), we obtain the conservation of mass, parallel momentum, and energy. Letting now $\psi = \ln f$, we obtain the decay of the entropy. Conversely, the only functions ψ for which $\int_{\mathbb{R}\times\mathbb{R}_+} Q(f, f)(V)\psi(V) v_\perp\, dV = 0$ for all f (or equivalently for which the vector $(\partial_\parallel \psi(V) - \partial_\parallel \psi(V'), \partial_\perp \psi(V), \partial_\perp \psi(V'))$ is colinear to the vector $(v_\parallel - v'_\parallel, v_\perp, v'_\perp)$ for all $V, V' \in \mathbb{R}\times\mathbb{R}_+$) are linear combinations of 1, $v_\parallel$, $v_\parallel^2 + v_\perp^2$. This fact can be proved as in the three-dimensional case [6].

The numerical approximation of the axisymmetric operator entails a symmetrization of the continuous formulation with respect to the axis ($v_\perp = 0$). The functions f and ψ are then extended to $\mathbb{R} \times \mathbb{R}$ to even functions with respect to $v_\perp$. Using this symmetrization, we obtain a numerical scheme satisfying the conservation of mass, parallel momentum, and energy, the decay of the entropy, and the fact that the only steady states are Maxwellians. The present numerical study treats the Coulombian case (of physical interest) as well as the Maxwellian case ($\gamma = 0$) which enables us to compare the numerical results with exact solutions [16].

Recently, conservative and entropy discretizations of the axisymmetric FPL operator were investigated in [11], but our approach is different and simpler: first the expression of the operator is explicitly computed (the integration with respect to the angle is carried out), and second, the spurious collisional invariants are eliminated in a very simple way and no perturbation process is needed. Furthermore, we show in Section 4 how the fast algorithms that were preformed in a 3D computation can be applied in the cylindrical geometry case. These fast algorithms dramatically reduce the numerical cost without destroying the physical properties of the scheme and its accuracy.

The paper is organized as follows: In Section 2 we present a simple expression of the axisymmetric FPL operator and give a symmetrized version with respect to the axis. In Section 3 we present a numerical discretization of the axisymmtric FPL operator that preserves all the physical properties described above at the discrete level. In Section 4 we show how to use fast algorithms without affecting the properties of the scheme and its accuracy. Finally, in Section 5 we give some numerical tests illustrating the efficiency of such discretizations and such algorithms.

2. EXPRESSIONS OF THE AXISYMMETRIC FPL OPERATOR

PROPOSITION 2.1. *If f only depends on $V = (v_\parallel, v_\perp)$, then*:

(i) *$Q(f, f)$ only depends on V.*

(ii) *There exists a 3×3 matrix $\Omega(V, V')$, symmetric, positive semi-definite, such that*

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}_+} Q(f,f)(V)\psi(V)v_\perp\,dV \\
&\quad = -\frac{1}{2}\iint_{(\mathbb{R}\times\mathbb{R}_+)^2} f(V)\,f(V')\begin{pmatrix} \partial_\parallel\psi(V) - \partial_\parallel\psi(V') \\ \partial_\perp\psi(V) \\ \partial_\perp\psi(V') \end{pmatrix}^T \Omega(V,V') \\
&\qquad \times \begin{pmatrix} \partial_\parallel(\ln f)(V) - \partial_\parallel(\ln f)(V') \\ \partial_\perp(\ln f)(V) \\ \partial_\perp(\ln f)(V') \end{pmatrix} v_\perp v'_\perp\,dV\,dV' \qquad (2.9)
\end{aligned}$$

for all test functions $\psi(V)$. The nullspace of $\Omega(V, V')$ is

$$\operatorname{Ker}\Omega(V,V') = \mathbb{R}\begin{pmatrix} v_\parallel - v'_\parallel \\ v_\perp \\ v'_\perp \end{pmatrix} \quad \text{if } (v_\parallel - v'_\parallel, v_\perp, v'_\perp) \neq (0,0,0). \qquad (2.10)$$

From these properties of $\Omega(V, V')$ *and the weak formulation* (2.9), *we deduce the following properties on the collisional operator*:

(a) *The collisional invariants are linear combination of mass, parallel momentum, and energy. That is, if* $\int_{\mathbb{R}\times\mathbb{R}_+} Q(f, f)(V)\psi(V)v_\perp\, dV = 0$, *for all* f *then* ψ *is a linear combination of* 1, $v_\parallel$ *and* $v_\parallel^2 + v_\perp^2$.

(b) *The H-theorem*,

$$\int_{\mathbb{R}\times\mathbb{R}_+} Q(f, f)(V)\ln f(V) v_\perp\, dV \le 0 \qquad \forall f, \tag{2.11}$$

the left-hand-side of (2.11) *is equal to* 0 *if and only if* f *is a Maxwellian*,

$$f(V) = \exp\bigl(C_1 + C_2 v_\parallel + C_3\bigl(v_\parallel^2 + v_\perp^2\bigr)\bigr)$$

with C_1, C_2, *and* C_3 *being real constants. This property translates the decay of the kinetic entropy* (*in the homogeneous case*),

$$\frac{d}{dt}\left(\int_{\mathbb{R}\times\mathbb{R}_+} f(V)\ln f(V) v_\perp\, dV\right) \le 0.$$

Proof. (i) This is a straightforward consequence of the invariance under orthogonal transformations of $\mathbb{R}^3$ of the FPL operator: $Q(f\circ R, f\circ R) = Q(f, f)\circ R$ for any orthogonal transformation R of $\mathbb{R}^3$.

(ii) By using the notations (1.8), the 3-dimensional gradient of an axisymmetric function ψ may be written as

$$\nabla\psi(V) = \partial_\parallel\psi(V) I + \partial_\perp\psi(V) U, \qquad \nabla\psi(V') = \partial_\parallel\psi(V') I + \partial_\perp\psi(V') U'. \tag{2.12}$$

Inserting these expressions in the 3-dimensional expression (1.3), and making the cylindrical change of variables in the integrals, we obtain the weak formulation (2.9). In particular, the matrix Ω is given by

$$\Omega(V, V') = \begin{pmatrix} A & B & -B' \\ B & C & -D \\ -B' & -D & C' \end{pmatrix} \tag{2.13}$$

with

$$\begin{cases} A = A(V, V') = \int_0^{2\pi}\int_0^{2\pi} I^T \Phi(v - v') I\, d\theta\, d\theta', \\ B = B(V, V') = \int_0^{2\pi}\int_0^{2\pi} U^T \Phi(v - v') I\, d\theta\, d\theta' \\ C = C(V, V') = \int_0^{2\pi}\int_0^{2\pi} U^T \Phi(v - v') U\, d\theta\, d\theta', \\ D = D(V, V') = \int_0^{2\pi}\int_0^{2\pi} U^T \Phi(v - v') U'\, d\theta\, d\theta', \end{cases} \tag{2.14}$$

where $x\cdot y$ denotes the scalar product in $\mathbb{R}^3$ between x and y. First we see that Ω is

symmetric. Simplified expressions of A, B, C, and D can be obtained after a few calculations,

$$\begin{cases} A = A(V, V') = 2\pi \int_0^{2\pi} \Lambda(V, V', \cos\eta)\big(v_\perp^2 + v_\perp'^2 - 2v_\perp v_\perp' \cos\eta\big)\, d\eta \\ B = B(V, V') = -2\pi \int_0^{2\pi} \Lambda(V, V', \cos\eta)(v_\parallel - v_\parallel')(v_\perp - v_\perp' \cos\eta)\, d\eta \\ C = C(V, V') = 2\pi \int_0^{2\pi} \Lambda(V, V', \cos\eta)\big[(v_\parallel - v_\parallel')^2 + v_\perp'^2 \sin^2\eta\big]\, d\eta \\ D = D(V, V') = 2\pi \int_0^{2\pi} \Lambda(V, V', \cos\eta)\big[(v_\parallel - v_\parallel')^2 \cos\eta + v_\perp v_\perp' \sin^2\eta\big]\, d\eta, \end{cases} \tag{2.15}$$

where

$$\Lambda(V, V', \cos\eta) = K\big(\big[(v_\parallel - v_\parallel')^2 + v_\perp^2 + v_\perp'^2 - 2v_\perp v_\perp' \cos\eta\big]^{1/2}\big) \tag{2.16}$$

and K is the function arising in the definition of the matrix Φ given by (1.2). A and D are symmetric with respect to V and V': $A(V, V') = A(V', V)$ and $D(V, V') = D(V', V)$. Instead, B and C are not symmetric and we set $B' = B(V', V)$ and $C' = C(V', V)$. Using these formulae, we easily check that

$$\begin{pmatrix} v_\parallel - v_\parallel' \\ v_\perp \\ v_\perp' \end{pmatrix} \in \operatorname{Ker} \Omega(V, V').$$

Furthermore, the rank of the matrix Ω is equal to 2 when $(v_\parallel - v_\parallel', v_\perp, v_\perp') \neq 0$. Indeed, by Cauchy–Schwartz inequality, we have $B^2 \leq AC$ and $D^2 \leq CC'$ and the equalities hold simultaneously if and only if $v_\parallel - v_\parallel' = v_\perp = v_\perp' = 0$. On the other hand, Ω can be viewed as the matrix representation of the bilinear form defined on $L^2([0, 2\pi]^2, \mathbb{R}^3)$ by

$$\mathcal{B}(u_1, u_2) = \langle \Phi(v - v')u_1, u_2 \rangle_{L^2([0,2\pi]^2, \mathbb{R}^3)} = \int_0^{2\pi} \int_0^{2\pi} [\Phi(v - v')u_1] \cdot u_2 \, d\theta \, d\theta'$$

in the basis (I, U, U'). The positivity of Ω then comes from the fact that $\Phi(v - v')$ is a 3×3 positive matrix.

Of course, the weak formulation (2.9) is equivalent to the following explicit expression of the collision operator,

$$Q(f, f)(V) = Q_\parallel(V) + Q_\perp(V) \tag{2.17}$$

with

$$\begin{aligned} Q_\parallel(V) = \partial_\parallel \int_{\mathbb{R}\times\mathbb{R}_+} &\{A[\partial_\parallel(\ln f)(V) - \partial_\parallel(\ln f)(V')] \\ &+ B\partial_\perp(\ln f)(V) - B'\partial_\perp(\ln f)(V')\} f(V) f(V') v_\perp' \, dV' \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} Q_\perp(V) = \frac{1}{v_\perp}\partial_\perp \int_{\mathbb{R}\times\mathbb{R}_+} &\{B[\partial_\parallel(\ln f)(V) - \partial_\parallel(\ln f)(V')] \\ &+ C\partial_\perp(\ln f)(V) - D\partial_\perp(\ln f)(V')\} f(V) f(V') v_\perp v_\perp' \, dV'. \end{aligned} \tag{2.19}$$

Now, in order to obtain discretizations of the axisymmetric FPL operator that preserve the conservation and entropy properties at the discrete level, we have to make sure that the

discretization keeps the symmetry between the variables V and V' that allows us to write a weak formulation similar to (2.9) at the discrete level. Yet one problem in a such a discretization is how to discretize near the axis under the constraints that all the physical properties described above have to be satisfied. To overcome this problem, we first extend the distribution function f to an even function with respect to the orthogonal component defined on $\mathbb{R}\times\mathbb{R}$ instead of $\mathbb{R}\times\mathbb{R}_+$. To preserve this property during the time evolution, we must have an even collision operator with respect to the orthogonal component. Observing that if $\psi(V)$ is even/odd with respect to $v_\perp$ then $\partial_\parallel\psi(V)$ is even/odd and $\partial_\perp\psi(V)$ is odd/even with respect to $v_\perp$, we obtain the following formulation in which the axis is not a boundary of the integration domain as in (2.9):

PROPOSITION 2.2. *If $f(V)$ is an even function with respect to $v_\perp$ then the extension of the collision operator $Q(f,f)$ defined by* (2.9) *to an even function* (*with respect to $v_\perp$*) *defined on $\mathbb{R}\times\mathbb{R}$ satifies, for all test function $\psi(V)$* (*even with respect to $v_\perp$*), *the following weak formulation*,

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} Q(f,f)(V)\psi(V)|v_\perp|\,dV&\\ =-\frac{1}{4}\int\int_{(\mathbb{R}\times\mathbb{R})^2} f(V)f(V')&\begin{pmatrix}\partial_\parallel\psi(V)-\partial_\parallel\psi(V')\\ \partial_\perp\psi(V)\\ \partial_\perp\psi(V')\end{pmatrix}^T\tilde\Omega(V,V')\\ \times&\begin{pmatrix}\partial_\parallel(\ln f)(V)-\partial_\parallel(\ln f)(V')\\ \partial_\perp(\ln f)(V)\\ \partial_\perp(\ln f)(V')\end{pmatrix}|v_\perp|\,|v'_\perp|\,dV\,dV'\end{aligned}\tag{2.20}$$

with

$$\tilde\Omega(V,V')=\begin{pmatrix}\tilde A & s(v_\perp)\tilde B & -s(v'_\perp)\tilde B'\\ s(v_\perp)\tilde B & \tilde C & -s(v_\perp)s(v'_\perp)\tilde D\\ -s(v'_\perp)\tilde B' & -s(v_\perp)s(v'_\perp)\tilde D & \tilde C'\end{pmatrix}.\tag{2.21}$$

$\tilde A$, $\tilde B$, $\tilde C$, $\tilde D$, $\tilde B'$, and $\tilde C'$ are the extension to $(\mathbb{R}\times\mathbb{R})^2$ of A, B, C, D, B', and C' to even functions with respect to $v_\perp$ and $v'_\perp$. $s(x)$ is the usual sign function of x.

The matrix $\tilde\Omega(V,V')$ is still symmetric, positive, and semi-definite. Its nullspace is the one-dimensional space spanned by $(v_\parallel-v'_\parallel, v_\perp, v'_\perp)$. Thus the new collision operator satisfies the same physical properties as in Proposition 2.1.

Now this symmetrized weak formulation is equivalent to the following explicit expression of the collision operator defined on all $\mathbb{R}\times\mathbb{R}$ instead of $\mathbb{R}\times\mathbb{R}_+$,

$$Q(f,f)(V)=Q_\parallel(V)+Q_\perp(V)\tag{2.22}$$

with

$$\begin{aligned}Q_\parallel(V)=\frac{1}{2}\partial_\parallel\int_{\mathbb{R}\times\mathbb{R}}&\{\tilde A[\partial_\parallel(\ln f)(V)-\partial_\parallel(\ln f)(V')]+s(v_\perp)\tilde B\partial_\perp(\ln f)(V)\\ &-s(v'_\perp)\tilde B'\partial_\perp(\ln f)(V')\}f(V)f(V')|v'_\perp|\,dV'\end{aligned}\tag{2.23}$$

and

$$Q_\perp(V) = \frac{1}{2|v_\perp|}\partial_\perp \int_{\mathbb{R}\times\mathbb{R}} \{s(v_\perp)\tilde{B}[\partial_\parallel(\ln f)(V) - \partial_\parallel(\ln f)(V')] + \tilde{C}\partial_\perp(\ln f)(V) - s(v_\perp)s(v'_\perp)\tilde{D}\partial_\perp(\ln f)(V')\} f(V) f(V') |v_\perp|\, |v'_\perp|\, dV'. \tag{2.24}$$

3. COMPLETELY CONSERVATIVE AND ENTROPIC DISCRETIZATIONS OF THE AXISYMMETRIC FPL OPERATOR

We consider a regular discretization of $\mathbb{R}\times\mathbb{R}$ of the form $V_i = (i_\parallel \Delta v, (i_\perp + \frac{1}{2})\Delta v)$ with $i = (i_\parallel, i_\perp) \in \mathbb{Z}\times\mathbb{Z}$, and $\Delta v > 0$. We will see that the "$\frac{1}{2}$ shift" in the discretization of the orthogonal velocity component allows us to satisfy the property (a) of Proposition 2.1 at the discrete level. We denote by $\bar{f}_i$ or $\bar{f}_{i_\parallel, i_\perp}$ an approximation of $f(V_i)$ and set $e_\parallel = (1, 0)$ and $e_\perp = (0, 1)$. Let D be a finite difference operator that approximates the axisymmetric gradient $(\partial_\parallel, \partial_\perp)$. We denote by $D_\parallel$ and $D_\perp$ the corresponding finite difference operator approximating $\partial_\parallel$ and $\partial_\perp$, i.e., $D = (D_\parallel, D_\perp)$. Let $D^* = (D_\parallel^*, D_\perp^*)$ the formal adjoint of D. Then the evenness condition of the discretised distribution function $\bar{f}$ on the velocity grid simply reads $\bar{f}_{i_\parallel, -i_\perp} = \bar{f}_{i_\parallel, i_\perp - 1} = \bar{f}_{i-e_\perp}$. The following result gives an approximation $\bar{Q}(\bar{f}, \bar{f})_i$ of $Q(f, f)(V_i)$ that satisfies the same condition of evenness as $\bar{f}$ and all the discrete analogues of the physical properties mentioned in Proposition 2.1.

PROPOSITION 3.1. *Let $D = (D_\parallel, D_\perp)$ and $D^* = (D_\parallel^*, D_\perp^*)$ be the two following finite difference operators*,

$$\begin{cases} (D_\parallel \bar{\psi})_i = \frac{\psi_{i+e_\parallel} - \psi_i}{\Delta v}, & (D_\perp \bar{\psi})_i = \frac{\psi_{i+e_\perp} - \psi_{i-e_\perp}}{2\Delta v} \\ (D_\parallel^* \bar{\psi})_i = \frac{\psi_{i-e_\parallel} - \psi_i}{\Delta v}, & (D_\perp^* \bar{\psi})_i = \frac{\psi_{i-e_\perp} - \psi_{i+e_\perp}}{2\Delta v}. \end{cases} \tag{3.25}$$

Then we can define an approximation $\bar{Q}(\bar{f}, \bar{f})_i$ of $Q(f, f)(V_i)$ which is even (*when $\bar{f}$ is even*) *with respect to the orthogonal component*,

$$\sum_{i\in\mathbb{Z}\times\mathbb{Z}} \left|i_\perp + \tfrac{1}{2}\right| \bar{Q}(\bar{f}, \bar{f})_i \bar{\psi}_i = -\frac{1}{4} \sum_{(i,j)\in(\mathbb{Z}\times\mathbb{Z})^2} \left|i_\perp + \tfrac{1}{2}\right| \left|j_\perp + \tfrac{1}{2}\right| \bar{f}_i \bar{f}_j \begin{pmatrix} (D_\parallel \bar{\psi})_i - (D_\parallel \bar{\psi})_j \\ (D_\perp \bar{\psi})_i \\ (D_\perp \bar{\psi})_j \end{pmatrix}^T \times \tilde{\Omega}(V_i, V_j) \begin{pmatrix} D_\parallel(\ln \bar{f})_i - D_\parallel(\ln \bar{f})_j \\ D_\perp(\ln \bar{f})_i \\ D_\perp(\ln \bar{f})_j \end{pmatrix} (\Delta v)^3. \tag{3.26}$$

Such a discretisation has the following properties:

(i) *Conservation of mass*, *parallel momentum*, *and energy*,

$$\sum_{i\in\mathbb{Z}\times\mathbb{Z}} \left|i_\perp + \tfrac{1}{2}\right| \bar{Q}(\bar{f}, \bar{f})_i \begin{pmatrix} 1 \\ i_\parallel \\ i_\parallel^2 + \left(i_\perp + \frac{1}{2}\right)^2 \end{pmatrix} = 0. \tag{3.27}$$

(ii) *Discrete H-theorem*,

$$\sum_{i\in\mathbb{Z}\times\mathbb{Z}} \big(i_\perp + \tfrac{1}{2}\big)\,\bar{Q}(\bar{f},\bar{f})_i(\ln \bar{f})_i \le 0. \tag{3.28}$$

(iii) *The collisional invariants* (*i.e.*, *the even sequences* $\bar{\psi}$ *such that* $\sum_{i\in\mathbb{Z}\times\mathbb{Z}} |i_\perp + \frac{1}{2}|\bar{Q}(\bar{f},\bar{f})_i\bar{\psi}_i = 0$ *for all sequence* $\bar{f}$) *are linear combinations of* 1, $i_\parallel$, *and* $i_\parallel^2 + (i_\perp + \frac{1}{2})^2$. *This is equivalent to saying that the only discrete steady states* (*i.e.*, *the sequences* $\bar{f}$ *such that* $\bar{Q}(\bar{f},\bar{f}) = 0$) *are the discrete Maxwellians* (*i.e.*, *the exponentials of linear combinations of the above three quantities*).

Proof. Using the finite difference operators given by (3.25), we define the approximation $\bar{Q}(\bar{f},\bar{f})_i$ of $Q(f,f)(V_i)$ by the explicit formulas

$$\bar{Q}_i(\bar{f},\bar{f}) = \bar{Q}_i^{\parallel} + \bar{Q}_i^{\perp} \tag{3.29}$$

with

$$\begin{aligned}\bar{Q}_i^{\parallel} = -\frac{1}{2}D_\parallel^* \sum_{j\in\mathbb{Z}} \big\{ &\tilde{A}(V_i,V_j)[D_\parallel(\ln \bar{f})_i - D_\parallel(\ln \bar{f})_j] + s\big(i_\perp+\tfrac{1}{2}\big)\tilde{B}(V_i,V_j)D_\perp(\ln f)_i \\ &- s\big(j_\perp+\tfrac{1}{2}\big)\tilde{B}(V_j,V_i)D_\perp(\ln f)_j\big\}\,\bar{f}_i\,\bar{f}_j\big|j_\perp+\tfrac{1}{2}\big|(\Delta v)^3\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\bar{Q}_i^{\perp} = -\frac{1}{2|i_\perp+1/2|}D_\perp^* \sum_{j\in\mathbb{Z}} \big\{ &s\big(i_\perp+\tfrac{1}{2}\big)\tilde{B}(V_i,V_j)[D_\parallel(\ln \bar{f})_i - D_\parallel(\ln \bar{f})_j] \\ &+ \tilde{C}(V_i,V_j)D_\perp(\ln f)_i - s\big(i_\perp+\tfrac{1}{2}\big)s\big(j_\perp+\tfrac{1}{2}\big)\tilde{D}(V_i,V_j)D_\perp(\ln f)_j\big\} \\ &\times \bar{f}_i\,\bar{f}_j\big|i_\perp+\tfrac{1}{2}\big|\,\big|j_\perp+\tfrac{1}{2}\big|(\Delta v)^3.\end{aligned} \tag{3.31}$$

As in the continuous case, this discrete collision operator is even with respect to the orthogonal component. This fact is due to the specific properties of the finite difference operator chosen above. Indeed, if $\bar{f}$ is even ($\bar{f}_{i_\parallel,-i_\perp} = \bar{f}_{i_\parallel,i_\perp-1}$) then $D_\parallel\bar{f}$ is even and $D_\perp\bar{f}$ is odd, that is, $(D_\parallel\bar{f})_{i_\parallel,-i_\perp} = (D_\parallel\bar{f})_{i_\parallel,i_\perp-1}$ and $(D_\perp\bar{f})_{i_\parallel,-i_\perp} = -(D_\perp\bar{f})_{i_\parallel,i_\perp-1}$. Now from (3.29) and by simple discrete integrations by parts, we obtain the weak formulation (3.26). The property (i) is then obtained by taking successively $\bar{\psi}_i = 1$, $i_\parallel$, and $i_\parallel^2 + (i_\perp+\frac{1}{2})^2$ in the weak formulation and using the nullspace of $\tilde{\Omega}$. In the same way, we let $\bar{\psi}_i = (\ln \bar{f})_i$ and get the discrete H-theorem thanks to the positivity of the matrix $\tilde{\Omega}$.

Now let $\bar{\psi}$ be a collisional invariant, that is, $\sum_{i\in\mathbb{Z}\times\mathbb{Z}} |i_\perp+\frac{1}{2}|\bar{Q}(\bar{f},\bar{f})_i\bar{\psi}_i = 0$ for all test sequence $\bar{f}$. Taking $\bar{f} = \exp(\bar{\psi})$ and using the positivity of $\tilde{\Omega}$ we obtain

$$\begin{pmatrix} (D_\parallel\bar{\psi})_i - (D_\parallel\bar{\psi})_j \\ (D_\perp\bar{\psi})_i \\ (D_\perp\bar{\psi})_j \end{pmatrix} \in \operatorname{Ker}\tilde{\Omega}(V_i,V_j) \qquad \text{for all } (i,j)\in(\mathbb{Z}\times\mathbb{Z})^2. \tag{3.32}$$

This implies the existence of reals λ_{ij} such that

$$\begin{cases} (D_\parallel\bar{\psi})_i - (D_\parallel\bar{\psi})_j = \lambda_{ij}(i_\parallel - j_\parallel), \\ (D_\perp\bar{\psi})_i = \lambda_{ij}\big(i_\perp+\frac{1}{2}\big) \\ (D_\perp\bar{\psi})_j = \lambda_{ij}\big(j_\perp+\frac{1}{2}\big) \end{cases} \tag{3.33}$$

for all $(i, j) \in (\mathbb{Z} \times \mathbb{Z})^2$. From the last two equalities we easily conclude that λ_{ij} is independent of i and j. Let $\lambda_{ij} = \lambda$, and we have

$$\begin{cases} (D_\parallel \bar{\psi})_i = \lambda i_\parallel + \alpha \\ (D_\perp \bar{\psi})_i = \lambda\big(i_\perp + \frac{1}{2}\big), \end{cases} \tag{3.34}$$

where λ and α are constants. Using the expressions of $D_\parallel$ and $D_\perp$ given by (3.25), we obtain

$$\begin{cases} \bar{\psi}_{i_\parallel, 2i_\perp} = \mu\big(i_\parallel^2 + \big(2i_\perp + \frac{1}{2}\big)^2\big) + \alpha i_\parallel + \beta_1 \\ \bar{\psi}_{i_\parallel, 2i_\perp + 1} = \mu\big(i_\parallel^2 + \big(2i_\perp + \frac{3}{2}\big)^2\big) + \alpha i_\parallel + \beta_2 \end{cases} \tag{3.35}$$

with μ, α, β_1, and β_2 being real constants. Now we write the evenness condition on the sequence $\bar{\psi}$, $\bar{\psi}_{i_\parallel, -i_\perp} = \bar{\psi}_{i_\parallel, i_\perp - 1}$, and then get $\beta_1 = \beta_2$. In this last evenness identification, we note the "$\frac{1}{2}$ shift" of the velocity grid in the orthogonal direction is of crucial importance. Without this shifting, spurious (non-physical) collisional invariants appear with the use of the centered finite difference operator in the orthogonal direction. Now if $\bar{f}$ is a steady state (i.e., $\bar{Q}(\bar{f}, \bar{f}) = 0$) then we deduce from (3.26) that $\ln \bar{f}$ is a collisional invariant and then $\bar{f}$ is a Maxwellian. This concludes the proof of (iii). ∎

When the function K arising in the expression of the coefficients A, B, C, and D (2.15) is given by $K(|v|) = |v|^\gamma$, the exact computation of these coefficients is not possible in general (in particular for the Coulombian case $\gamma = -3$) because it involves elliptic integrals. On the other hand, a numerical approximation of such integrals will be very expensive because this would have to be done for each point of the velocity grid and at each time step. That is why we choose to approximate these integrals by series expansions using exact computations of trigonometrical integrals. We first set

$$a = (v_\parallel - v'_\parallel)^2 + v_\perp^2 + v'^2_\perp \text{ et } b = -2v_\perp v'_\perp$$

and expand (2.16) as

$$\begin{aligned} \Lambda(V, V', \cos\eta) &= K(|v - v'|) = |v - v'|^\gamma = |a + b\cos\eta|^{\gamma/2} \\ &= |a|^{\gamma/2} \sum_{k=0}^{+\infty} \frac{(\gamma/2)(\gamma/2 - 1)\cdots(\gamma/2 - k + 1)}{k!} \left(\frac{b}{a}\cos\eta\right)^k. \end{aligned}$$

We then insert this expansion in the expressions of $A(V, V')$, $B(V, V')$, $B(V, V')$, and $D(V, V')$ and obtain for $A(V, V')$ (for instance) the expansion

$$A(V, V') = 2\pi |a|^{\gamma/2} \left(v_\perp^2 + v'^2_\perp + \sum_{k=1}^{+\infty} \big((v_\perp^2 + v'^2_\perp)\gamma_k + a\gamma_{k-1} \big) \left(\frac{b}{a}\right)^k \int_0^{2\pi} \cos^k \eta \, d\eta \right)$$

with $\gamma_0 = 1$ and $\gamma_k = (\gamma/2)(\gamma/2 - 1)\cdots(\gamma/2 - k + 1)/k!$ for all $k \geq 1$. The odd terms of this expansions vanish and the even terms are given by the following Wallis explicit formula,

$$\int_0^{2\pi} \cos^{2k} \eta \, d\eta = 2\pi \frac{1 \cdot 3 \cdots (2k - 1)}{2 \cdot 4 \cdots 2k}.$$

An approximation is then obtained for the coefficients of the matrix Ω by simply truncating these expansions (the two or three first terms are sufficient in general). This approximation does not affect the conservation properties. Indeed, a simple computation shows that the vector $(v_\parallel - v'_\parallel, v_\perp, v'_\perp))$ remains in the nullspace of $\Omega(V, V')$ if we replace the function $\Lambda(V, V', \cos\eta)$ in (2.15) and (2.16) by any other quantity (in particuler by its expansion).

4. FAST ALGORITHMS

Our aim in this section is to reduce the numerical complexity (and then the computational cost) of the axisymmetric FPL operator. Precisely, we will show that the fast algorithms developed for the 3-dimensional operator [5, 15] can also be applied in a cylindrical geometry.

4.1. Sub-lattices method. Let a be an integer ($a \geq 2$), and let us define the following approximation of the axisymmetric Landau operator on the same velocity grid as defined above,

$$\bar{Q}_i[a] = \bar{Q}_i^{\parallel}[a] + \bar{Q}_i^{\perp}[a] \tag{4.36}$$

with

$$\begin{aligned}\bar{Q}_i^{\parallel}[a] = -\frac{1}{2} D_\parallel^* \sum_{j \equiv i[a]} \big\{ &\tilde{A}(V_i, V_j)[D_\parallel(\ln \bar{f})_i - D_\parallel(\ln \bar{f})_j] + s\big(i_\perp + \tfrac{1}{2}\big)\tilde{B}(V_i, V_j) D_\perp(\ln f)_i \\ &- s\big(j_\perp + \tfrac{1}{2}\big)\tilde{B}(V_j, V_i) D_\perp(\ln f)_j \big\} \bar{f}_i \bar{f}_j \big| j_\perp + \tfrac{1}{2}\big| (a\Delta v)^3 \end{aligned} \tag{4.37}$$

and

$$\begin{aligned}\bar{Q}_i^{\perp}[a] = -\frac{1}{2\big|i_\perp + \frac{1}{2}\big|} D_\perp^* \sum_{j \equiv i[a]} \big\{ &s\big(i_\perp + \tfrac{1}{2}\big)\tilde{B}(V_i, V_j)[D_\parallel(\ln \bar{f})_i - D_\parallel(\ln \bar{f})_j] \\ &+ \tilde{C}(V_i, V_j) D_\perp(\ln f)_i - s\big(i_\perp + \tfrac{1}{2}\big) s\big(j_\perp + \tfrac{1}{2}\big)\tilde{D}(V_i, V_j) D_\perp(\ln f)_j \big\} \\ &\times \bar{f}_i \bar{f}_j \big| i_\perp + \tfrac{1}{2}\big| \, \big| j_\perp + \tfrac{1}{2}\big| (a\Delta v)^3, \end{aligned} \tag{4.38}$$

where $j \equiv i[a]$ means that the $i - j$ components are a multiple of a. This new discrete operator is even when $\bar{f}$ is even with respect to the orthogonal component, and again satisfies the weak formulation

$$\begin{aligned}\sum_{i \in \mathbb{Z}\times\mathbb{Z}} \big| i_\perp + \tfrac{1}{2}\big| \bar{Q}_i[a]\bar{\psi}_i = -\frac{1}{4} \sum_{i,j, i \equiv j[a]} \big| i_\perp + \tfrac{1}{2}\big| \, \big| j_\perp + \tfrac{1}{2}\big| \bar{f}_i \bar{f}_j \begin{pmatrix} (D_\parallel \bar{\psi})_i - (D_\parallel \bar{\psi})_j \\ (D_\perp \bar{\psi})_i \\ (D_\perp \bar{\psi})_j \end{pmatrix}^T \\ \times \tilde{\Omega}(V_i, V_j) \begin{pmatrix} D_\parallel(\ln \bar{f})_i - D_\parallel(\ln \bar{f})_j \\ D_\perp(\ln \bar{f})_i \\ D_\perp(\ln \bar{f})_j \end{pmatrix} (a\Delta v)^3. \end{aligned} \tag{4.39}$$

Thus, this discretization satisfies the properties (i) and (ii) of Proposition 3.1 (conservation and entropy). However, and as in the 3-dimensional case [5], the property (iii) of

Proposition 3.1 is not satisfied. Non-physical collisional invariants (and then non-Maxwellian steady states) can be generated by this approximation (see [5] for details) . To solve this problem, we introduce the following new approximation:

PROPOSITION 4.1. *Let a and b be two mutually prime integers (for instance $b = a + 1$), and let $\bar{Q}_i[a, b]$ be the following approximation of the axisymmetric FPL operator*

$$\bar{Q}_i[a, b] = \frac{1}{2}(\bar{Q}_i[a] + \bar{Q}_i[b]). \tag{4.40}$$

Then $\bar{Q}_i[a, b]$ satifises the properties (i), (ii), and (iii) of Proposition 3.1.

Proof. The properties (i) and (ii) are straightforward. We prove the property (iii). Let $\bar{\psi}$ be a collisional invariant corresponding to $\bar{Q}[a, b]$. Then there exist reals λ_{ij} such that

$$\begin{cases} (D_\parallel \bar{\psi})_i - (D_\parallel \bar{\psi})_j = \lambda_{ij}(i_\parallel - j_\parallel), \\ (D_\perp \bar{\psi})_i = \lambda_{ij}\left(i_\perp + \frac{1}{2}\right) \\ (D_\perp \bar{\psi})_j = \lambda_{ij}\left(j_\perp + \frac{1}{2}\right) \end{cases} \tag{4.41}$$

for all $(i, j) \in (\mathbb{Z} \times \mathbb{Z})^2$ such that $i \equiv j[a]$ or $i \equiv j[b]$. From the last two equalities we easily conclude that λ_{ij} is independent of i and j. Let $\lambda_{ij} = \lambda$. We have

$$\begin{cases} (D_\parallel \bar{\psi})_i - (D_\parallel \bar{\psi})_{i-ka} = \lambda a k_\parallel, \\ (D_\parallel \bar{\psi})_{i-ka} - (D_\parallel \bar{\psi})_{i-ka-lb} = \lambda b l_\parallel, \\ (D_\perp \bar{\psi})_i = \lambda\left(i_\perp + \frac{1}{2}\right) \end{cases} \tag{4.42}$$

for all $(i, k, l) \in (\mathbb{Z} \times \mathbb{Z})^3$. Now remember that a and b are chosen to be mutually prime. Then by the Bezout identity, for all $(i, j) \in (\mathbb{Z} \times \mathbb{Z})^2$ we can find $(k, l) \in (\mathbb{Z} \times \mathbb{Z})^2$ such that $i - j = ak + bl$. We then sum the first two equalities of (4.42) and obtain

$$\begin{cases} (D_\parallel \bar{\psi})_i - (D_\parallel \bar{\psi})_j = \lambda a(i_\parallel - j_\parallel), \\ (D_\perp \bar{\psi})_i = \lambda\left(i_\perp + \frac{1}{2}\right) \end{cases} \tag{4.43}$$

for all $(i, j) \in (\mathbb{Z} \times \mathbb{Z})^2$. The situation is now the same as in the proof of (3.1). We then conclude in the same way. ∎

The interest of the sub-lattices method is of course its reduced cost. Instead of the quadratic cost of the order N^2, the sub-lattices strategy requires only $N^2/a^2 + N^2/b^2$ where N is the total number of the regular velocity points in the discretized domain. In general we choose $b = a + 1$ and use alternatively a or b at each time step. When N is of the order of 1000 points, the integer a can take the values $a = 7$ or 8 and then the computational cost is divided by a factor of the order of 50.

4.2. Multigrid Monte Carlo method. In this section we present an algorithm which reduces the complexity of the axisymmetric FPL operator (fromula (2.9)) from $O(N^2)$ to $O(N \ln N)$. This algorithm was already used in 3D computations [5]. Here we only do a brief presentation in a particular case of cylindrical symmetry and refer the readers to [5] for more details.

For simplifications, let us assume that the domain of the integration is a square box of length 1: $\mathcal{C}_0 = [0, 1] \times [-1/2, 1/2]$. The multigrid strategy consists on splitting $\mathcal{C}_0$ in several parts according to the following hierarchy.

At *Level one*, we split $\mathcal{C}_0$ into 4 equal square boxes (called its children) $\mathcal{C}_1^r$ $(r = 1 \ldots 4)$ and $\mathcal{C}_0$ is called their father. At *Level two*, we split each $\mathcal{C}_1^r$ into 4 equal square boxes (its children). and obtain 16 square boxes still denoted by $\mathcal{C}_2^r$ $(r = 1 \ldots 16)$, etc. The center of a square box $\mathcal{C}_k^r$ (of level k) is denoted by θ_k^r. We iterate this process until the finest mesh level *ng*. Let $\mathcal{C}_k^r$ and $\mathcal{C}_k^{r'}$ be two square boxes of level k. We say that $\mathcal{C}_k^r$ and $\mathcal{C}_k^{r'}$ are *well separated* if they are separated at least by one square box of the same level, else we say that they are *neighbors*. We also define and denote by Int $(\mathcal{C}_k^r)$ the *interaction list* of a given square box $\mathcal{C}_k^r$ of level k, as the set of square boxes $\mathcal{C}_k^{r'}$ of the same level k, which are well-separated, and whose fathers are neighbors. Thus following this process, the weak formulation may be written as

$$\int_{\mathcal{C}_0} Q(f, f)(V)\psi(V)|v_\perp|\, dV$$

$$= -\frac{1}{4}\sum_{k=1}^{+\infty} \sum_{r,r'/\mathcal{C}_k^{r'} \in \mathrm{Int}(\mathcal{C}_k^r)} \int\int_{\mathcal{C}_k^r \times \mathcal{C}_k^{r'}} f(V) f(V') \begin{pmatrix} \partial_\parallel \psi(V) - \partial_\parallel \psi(V') \\ \partial_\perp \psi(V) \\ \partial_\perp \psi(V') \end{pmatrix}^T \tilde{\Omega}(V, V')$$

$$\times \begin{pmatrix} \partial_\parallel (\ln f)(V) - \partial_\parallel (\ln f)(V') \\ \partial_\perp (\ln f)(V) \\ \partial_\perp (\ln f)(V') \end{pmatrix} |v_\perp|\, |v'_\perp|\, dV\, dV'. \tag{4.44}$$

Now to discretize this weak formula, we introduce a regular discretization of $\mathcal{C}_0$ in $N = 4^n$ velocity points V_i and use a Monte Carlo integration to evaluate the integrals on well separated square boxes,

$$\int\int_{\mathcal{C}_k^r \times \mathcal{C}_k^{r'}} F(V, V')\, dV\, dV' \sim 4^{n-k} \sum_{i \in \mathcal{C}_k^r} F\big(V_i, V_{\sigma(i)}\big)(\Delta v)^4, \tag{4.45}$$

where 4^{n-k} is equal to the number of velocity points of C_k^r (or $C_k^{r'}$) and σ is a random correspondence from C_k^r to $C_k^{r'}$. This Monte Carlo integration permits the reduction of the total computational cost from N^2 to $N \ln N$.

4.3. Multipole expansions. This strategy is deterministic and is an alternative method to the multigrid Monte Carlo method. We use the multigrid hierarchy to obtain integrations on only well separated sets as in (4.44). Here we do not approximate these integrals by Monte Carlo formulas as in (4.45) but use deterministic approximation based on multipole expansions. In the sequel we explain briefly how the method can be applied to the axisymmetric FPL operator and refer the readers to [15, 12] for more technical details of the method. As for the 3-dimensional case [15], the main observation is that the velocities V and V' in (4.44) are coupled only through the modulus of the relative velocity $\Lambda(V, V', \cos\eta) = [(v_\parallel - v'_\parallel)^2 + v_\perp^2 + v_\perp'^2 - 2v_\perp v'_\perp \cos\eta]^{\gamma/2}$ that appears in the expression of the matrix $\tilde{\Omega}(V, V')$, otherwise, the double integrals in (4.44) can be written exactly as a product (or sum of products) of simple ones and this fact reduces dramatically the complexity of the operator. This is the case of Maxwellian molecules where $\gamma = 0$. For

$\gamma \neq 0$, we see that when $V \in C_k^r$ and $V' \in C_k^{r'}$ and when C_k^r and $C_k^{r'}$ are well separated then we can approximate $\Lambda(V, V', \cos\eta)$ by its value $\Lambda(V_0, V_0', \cos\eta)$ at the centers V_0 and V_0' of C_k^r and $C_k^{r'}$; this is called *a multipole approximation at the* 0 *order* (see [15] for details). Thus all integrals on $C_k^r \times C_k^{r'}$ arising in (4.44) can be written as a sum of a products of simple integrals after such an approximation. The complexity of the formula (4.44) is then seriously reduced. If one wants more accuracy then the relative velocity has to be approximated by an expansion about its value $\Lambda(V_0, V_0', \cos\eta)$ at the centers of C_k^r and $C_k^{r'}$; this is called *multipole expansions*. In this paper, we only consider a multipole approximation at the 0 order. The numerical tests presented at the end of the paper show that the order 0 suffices for a good approximation of the FPL operator. Note that the multipole approximations only concern the term $\Lambda(V, V', \cos\eta)$; all the other terms in the matrix $\tilde{\Omega}(V, V')$ are replaced by their exact value at the points of the velocity grid. At this level of the application of multipole expansions, the total cost is reduced to $O(N \ln N)$ which is the same as for the Monte Carlo method. The use of multipole approximations presents at least two main advantages: the deterministic character of the strategy and its high accuracy. Indeed, the order of the multipole expansions enables us to control the error of the method and a very small order suffices to ensure a good accuracy. Note that only the order 0 suffices to obtain almost the same accuracy as multigrid Monte Carlo methods. In the multipole expansions, the approximation concerns only the relative velocity while all the terms under the integrals (4.44) are approximated in Monte Carlo integrations; thus, for the particular case of maxwellian potential ($\gamma = 0$), no approximation is needed and the multipole method has the same accuracy as the quadratic discretization and has a linear complexity.

We recall that the complete use of the fast multipole method (FMM) introduced by Greengard and Rokhlin [12] would give a complexity of the order CN instead of $N \ln N$. But the constant C is big and therfore the FMM method is only interesting for sufficiently large N. On the other hand, the use of an explicit in time scheme to solve the homogeneous FPL equation leads to a CFL condition that does not allow large values of N. This CFL condition is explicitly established in the particular case of isotropic distribution functions [4]. Thus, an interesting open problem is to find an implicit scheme to solve the homogenous FPL equation which would be conservative, entropic, and non-expensive.

5. SOME NUMERICAL RESULTS

We choose the velocity domain of the form $\mathcal{C}_0 = [0, v\max] \times [-v\max/2, v\max/2]$ and consider a regular discretization in N points of this domain. As in [6], the weak formulation (3.26) of the FPL operator can also be written for a bounded domain. The details about this point are not given in this paper since the reduction to a bounded domain is performed in the same way as in [6]. We take $N = (2n+1) \times 2n$. Precisely, the points of the velocity grid are $V_i = (i_\parallel, i_\perp + \frac{1}{2})\Delta v$, with $0 \leq i_\parallel \leq 2n$ and $-n \leq i_\perp \leq n-1$. The velocity step is then $\Delta v = v\max/2n$. In all our numerical tests, we deal with a time-explicit scheme to solve the homogeneous problem. The time step is computed at each iteration in time such that the positivity of the distribution function f is preserved. We shall consider the time evolutions (under only collisional effects) of the following quantities:

- Discrete kinetic entropy,

$$H_d(t) = \sum_{i \in \mathbb{Z}\times\mathbb{N}} \left(i_\perp + \tfrac{1}{2}\right) \bar{f}_i(t) \log \bar{f}_i(t)\, \Delta v^3. \tag{5.46}$$

- Order 4 moment,

$$\mathcal{M}_d^{(4)}(t) = \sum_{i \in \times\mathbb{N}} \left(i_\perp + \tfrac{1}{2}\right)(|i|\Delta v)^4 \, \bar{f}_i(t) \Delta v^3. \tag{5.47}$$

- Discrete mass, parallel momentum and energy,
  —mass

$$\mathcal{N}(t) = \sum_{i \in \mathbb{Z}\times\mathbb{N}} \left(i_\perp + \tfrac{1}{2}\right) \bar{f}_i(t) \, \Delta v^3, \tag{5.48}$$

  —parallel momentum (or mean velocity)

$$i_{0\|}\Delta v = \frac{1}{\mathcal{N}} \sum_{i \in \mathbb{Z}\times\mathbb{N}} \left(i_\perp + \tfrac{1}{2}\right) i_\| \, \bar{f}_i(t) \, \Delta v^4, \tag{5.49}$$

  —energy

$$E_d(t) = \sum_{i \in \times\mathbb{N}} \left(i_\perp + \tfrac{1}{2}\right)(|i|\Delta v)^2 \, \bar{f}_i(t) \, \Delta v^3. \tag{5.50}$$

- Discrete temperatures,

$$T_\|(t) = \sum_{i \in \mathbb{Z}\times\mathbb{N}} \left(i_\perp + \tfrac{1}{2}\right)[(i_\| - i_{0\|})\Delta v]^2 \, \bar{f}_i(t) \, \Delta v^3 \tag{5.51}$$

$$T_\perp(t) = \sum_{i \in \mathbb{Z}\times\mathbb{N}} \left(i_\perp + \tfrac{1}{2}\right)\left[\left(i_\perp + \tfrac{1}{2}\right)\Delta v\right]^2 \bar{f}_i(t) \, \Delta v^3 \tag{5.52}$$

$$T(t) = \frac{1}{3}(T_\|(t) + 2T_\perp(t)). \tag{5.53}$$

- Relative quadratic error. If f^{exact} is an exact solution in the Maxwellian case [16], and f is an approximate solution obtained by a numerical simulation starting from the initial data $f_0(v) = f^{exact}(0, v)$, then we define the quadratic error as

$$EQ(t) = \frac{\sum_{i \in \mathbb{Z}^3} \left| \bar{f}_i(t) - \bar{f}_i^{exact}(t) \right|^2}{\sum_{i \in \mathbb{Z}^3} f_i^{exact}(t)^2}. \tag{5.54}$$

The following numerical tests will concern both the Maxwellian case where the results are compared with an analytic solution and the Coulombian case. The three fast algorithms will be tested and compared: the sublattices method presented in Subsection 4.1, the multigrid Monte Carlo method presented in Subsection 4.2, and the multigrid multipole approximations presented in Subsection 4.3.

5.1. Maxwellian case ($\gamma = 0$). We take the initial distribution function in the class of isotropic exact solutions of the homogeneous FPL equation found in [16]. The simplest one is the following function which is isotropic about the mean velocity v_0,

$$f^{exact}(v, t) = M_{\mathcal{N}, v_0, T}(v)(1 + c_2 Q_2[(v - v_0)/v_{th}] \exp(-8 \times \mathcal{N} \times t)) \tag{5.55}$$

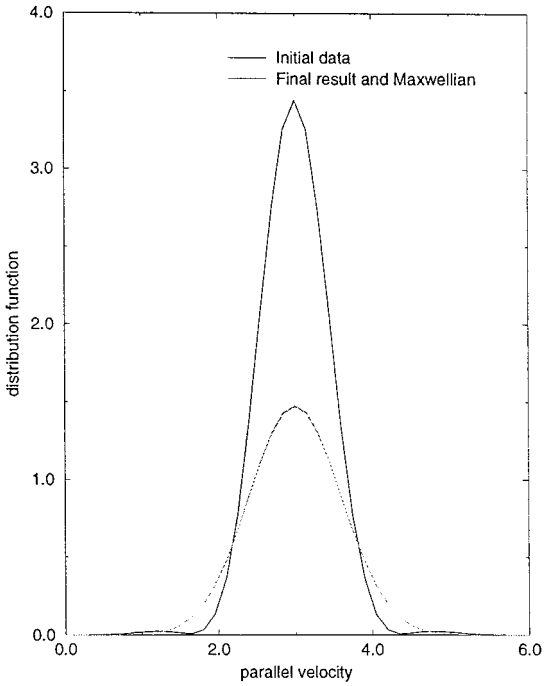

FIG. 1. Initial data $f_0(v_\parallel, 0)$ and equilibrium state in the Maxwellian case.

with

$$Q_2(v) = \frac{1}{120}(|v|^4 - 10|v|^2 + 15). \tag{5.56}$$

$M_{\mathcal{N},v_0,T}(v)$ is defined by (1.5) with $v_0 = (v\max/2, 0)$, $v_{th} = 0.6$, $\mathcal{N} = 5$, and $c_2 = 11$. We take $v\max = 6$. As the mean velocity v_0 has no orthogonal component; this initial distribution function is axisymmetric.

In Fig. 1 we can see that the final distribution function (obtained by the three numerical algorithms of Section 4) coincides with the Maxwellian that has the same parameters as the initial data. This shows equivalently that the only collisional invariants are linear combinations of the mass, the parallel momentum, and the energy. In Fig. 2, we have plotted the evolution in time of the entropy using the three algorithms together with the exact evolution . We see that we obtain a good appoximation of the entropy with a little bit more accurate results for the sublattices and multipole methods. The relative quadratic errors between the exact analytic solution and the approximate solution are plotted in Figs. 3 and 4. We can also see that the sublattices and mutipole methods are more accurate than the Monte Carlo method with a comparable computational cost (in terms of CPU time, see Table I). Figures 1–4 are performed with $N = 33 \times 16$ velocity points. In Fig. 5 we give the evolution in time of the relative quadratic errors for various velocity paths (or various values of N). In Fig. 6 the evolutions in time of the order 4 moment makes clear the advantage of the sublattices and multipole methods. Notice however that the relative variation of this moment is small. In all the numerical tests of this paper, we use a time-explicit scheme. In Table I, we give the CPU time per iteration (in time) required by the three algorithms and compare them with the computational cost of the quadratic axisymmetric scheme (discretization of

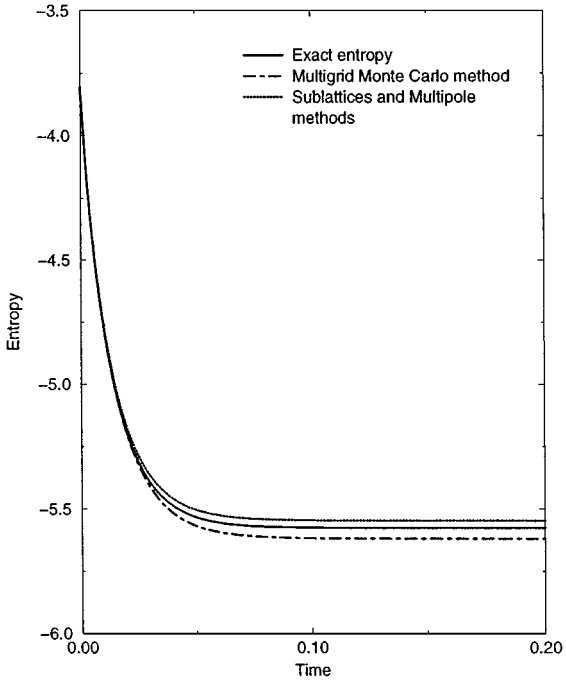

FIG. 2. Kinetic entropy in the Maxwellian case.

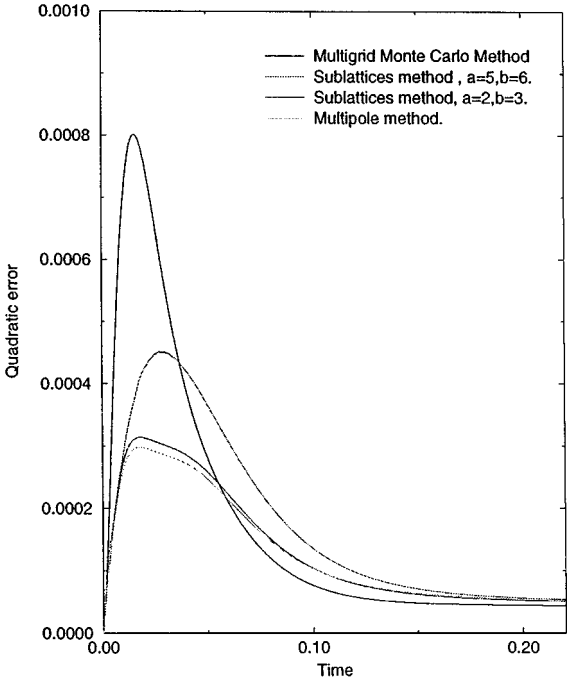

FIG. 3. Quadratic errors in the Maxwellian case.

TABLE I

N	17×8	33×16			65×32	
Sublattices sizes	2, 3	2, 3	3, 4	5, 6	7, 8	10, 11
Sublattices	0.004 s	0.04 s	0.02 s	0.01 s	0.1 s	0.04 s
Multigrids MC	0.008 s	0.04 s			0.27 s	
Multipole	0.001 s	0.003 s			0.008 s	
Quadratic schemes	0.009 s	0.1 s			3.6 s	

Proposition 3.1). These CPU times are carried out on the Computer DEC AlphaServeur 2100 4/275 OSF/1(Digital UNIX). Note that the computational costs of the multipole method are smaller but this is only due to the particular form of the collision operator in the Maxwellian case. In fact, the multipole method is of linear complexity in this case and has the same accuracy as quadratic schemes. We shall see that in the Coulombian case the computational cost of the zero-order multipole method is almost equivalent to those of the two other methods.

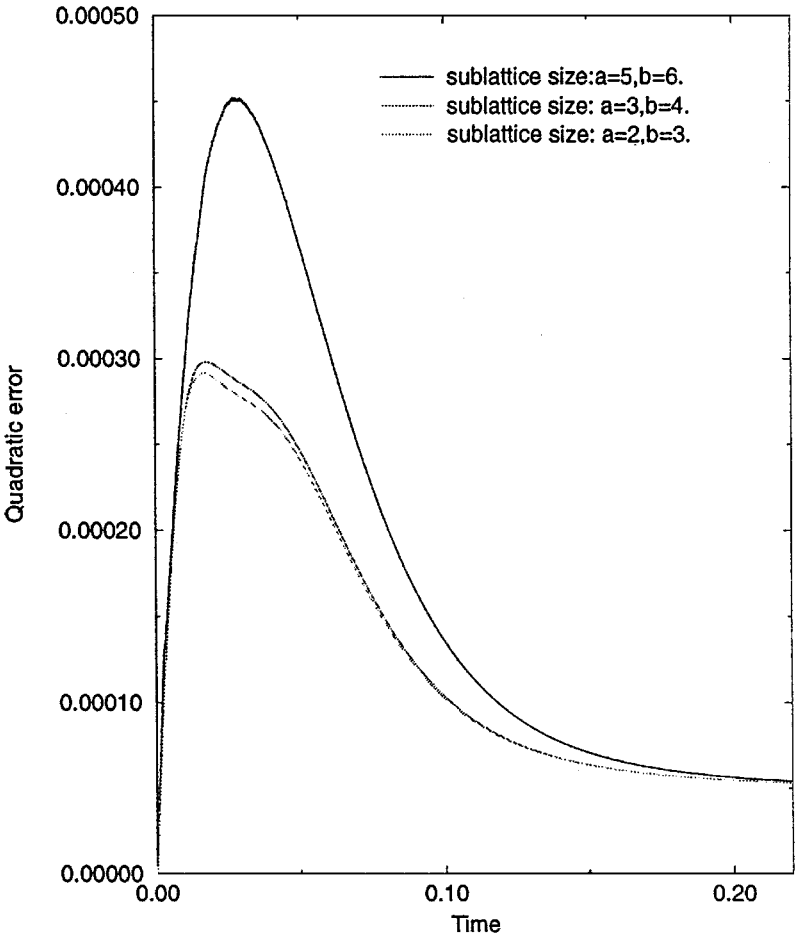

FIG. 4. Quadratic errors for various sizes of sublattices.

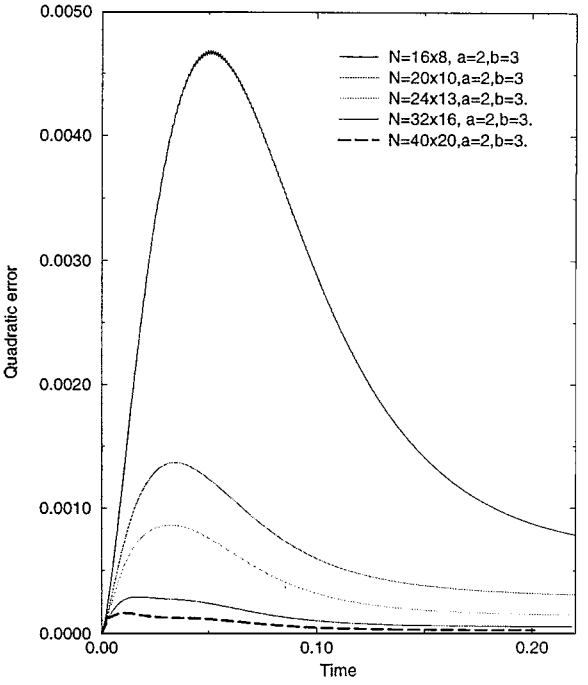

FIG. 5. Quadratic errors for various values of Δv.

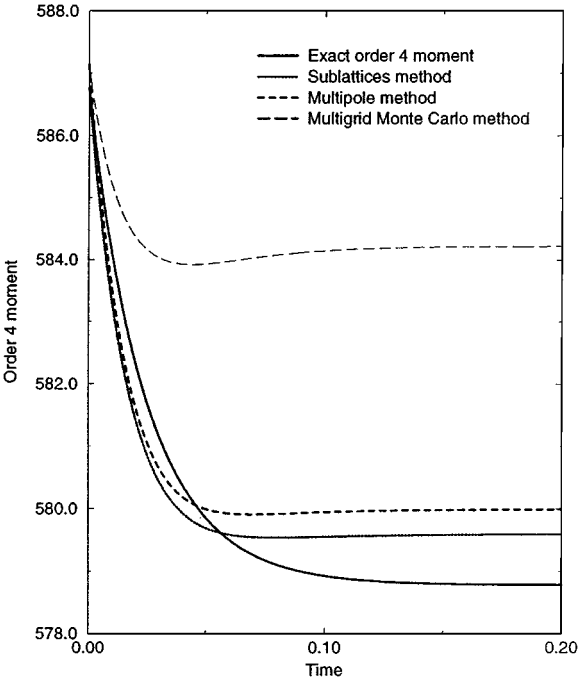

FIG. 6. Order 4 moment in the Maxwellian case.

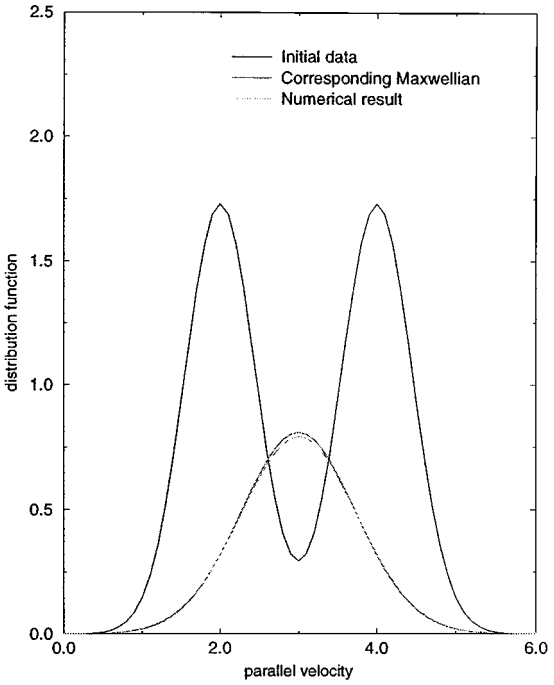

FIG. 7. Initial data $f_0(v_\parallel, 0)$, final distribution, and Maxwellian function in the Coulombian case.

5.2. Coulombian case ($\gamma = -3$). In this case, we do not know any explicit solution of the homogenous FPL equation. The initial data are chosen to be a bimaxwellian, that is,

$$f_0(v) = \frac{1}{2}(M_{\mathcal{N},v_{01},T}(v) + M_{\mathcal{N},v_{02},T}(v)), \tag{5.57}$$

where $M_{\mathcal{N},u,T}$ is given by (1.5), and $v_{01} = (2, 0)$, $v_{02} = (4, 0)$. Again the two velocities v_{01} and v_{02} have no orthogonal component and then the initial data are axisymmetric. Finally we take $v_{th} = 0.45$ (thermal velocity) and $\mathcal{N} = 5$ (density number of particles).

In Fig. 7, again we see that the final numerical distribution function coincides with the Maxwellian whose parameters (mass, mean velocity, and temperature) are the macroscopic quantities determined from the initial distribution . In Table II we give the maximum relative error (with respect to their initial value) on the mass, the parallel momentum, and the energy.

In Fig. 8, we can see the decay of the entropy and again observe that the Monte Carlo method relaxes to a slightly different state. The anisotropy of the initial data (about the mean velocity) enables us to see the relaxations of the temperatures in parallel and orthogonal

TABLE II

	Mass	Momentum	Energy
Sublattices	1.1360647E-06	3.0174860E-07	1.7855102E-06
Multigrids MC	1.0131008E-06	8.9009598E-07	1.1824987E-06
Multipole	1.7978964E-06	3.6916401E-07	1.0708056E-06
Quadratic schemes	1.1360647E-06	3.4645211E-07	1.9997715E-06

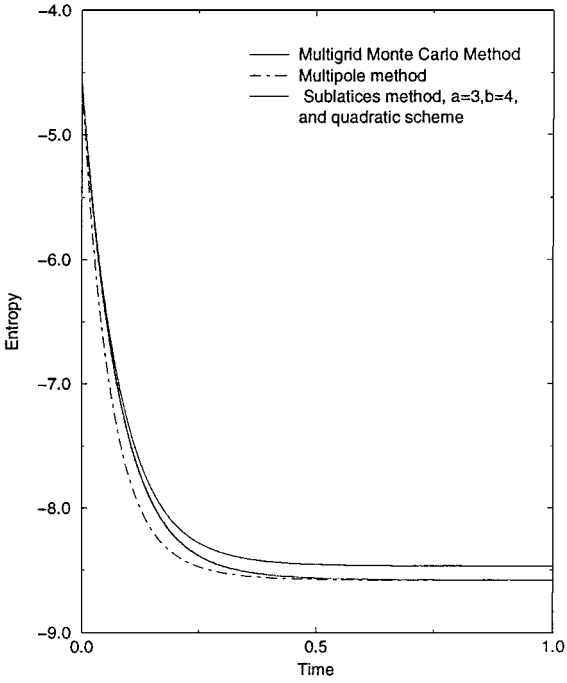

FIG. 8. Kinetic entropy in the Coulombian case.

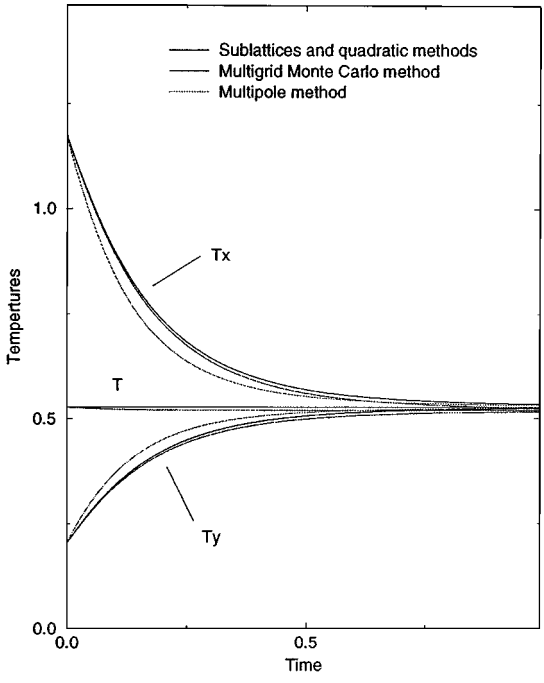

FIG. 9. Temperatures $T_x = T_\parallel$, $T_y = T_\perp$, and T in the Coulombian case.

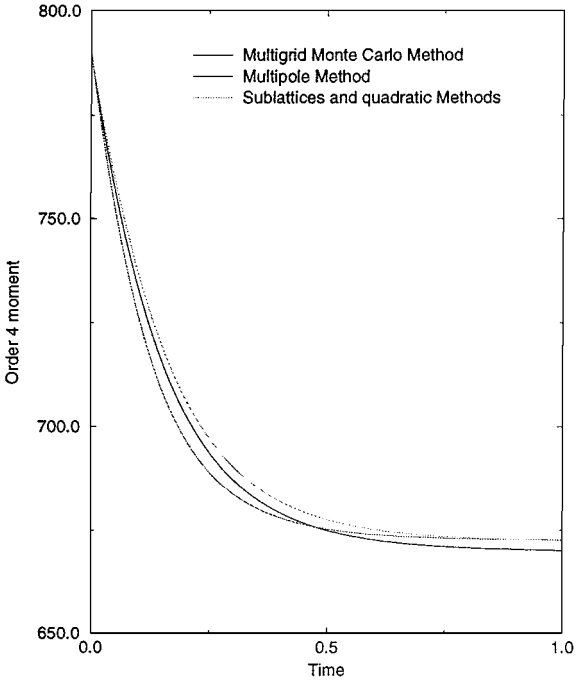

FIG. 10. Order 4 moment in the Coulombian case.

directions to the temperature of the Maxwellian equilibrium state (which is isotropic about the mean velocity); see Fig. 9. In these last two tests and also in Fig. 10 we remark that the mutipole method is a little bit less accurate than the two other methods (with the quadratic scheme as a reference), but converges to the the right value. The increase of the accuracy of the multipole method certainly needs mutipole expansions of higher orders but this would naturally increase the computational cost of the method. The same observation can be done in Fig. 10. Now, in order to illustrate the need of two sublattice sizes to eliminate spurious collisional invariants (see Subsection 4.1), we plot in Fig. 11 two relaxations of the value of f at the center of the grid. The first one is obtained by using only one sublattice size $a = 6$, and the second uses two mutually prime integers $a = 6$ and $b = 7$. We then see that the first curve does not converge to the value of the Maxwellain at the center of the grid. Figures 7–11 are performed on a velocity grid with $N = 33 \times 16$ points.

The last test concerns the initial data

$$f_0(v) = 0.01 \exp\{-10[(|v| - 0.3)/0.3]^2\}$$

which was tested in [18, 11]. The scheme in [18] leads to a final distribution function which is different from the right Maxwellian. The scheme in [11] is conservative entropic and converges to the right Maxwellian, but the computational cost is big (quadratic complexity) and a perturbative process is needed (to obtain the right equilibrium). Here we have tested this initial data on a grid with $N = 65 \times 32$ velocity points (see Fig. 12). We use a sublattice method with $a = 7$ and $b = 8$ and obtain a conservative and entropic relaxation to the right Maxwellian. In Fig. 12 we have plotted the distribution function at different time steps (ni denotes the number of time-iterations) and can see the convergence to the right Maxwellian. The computational cost of this simulation is divided by a factor of the order of 50 compared

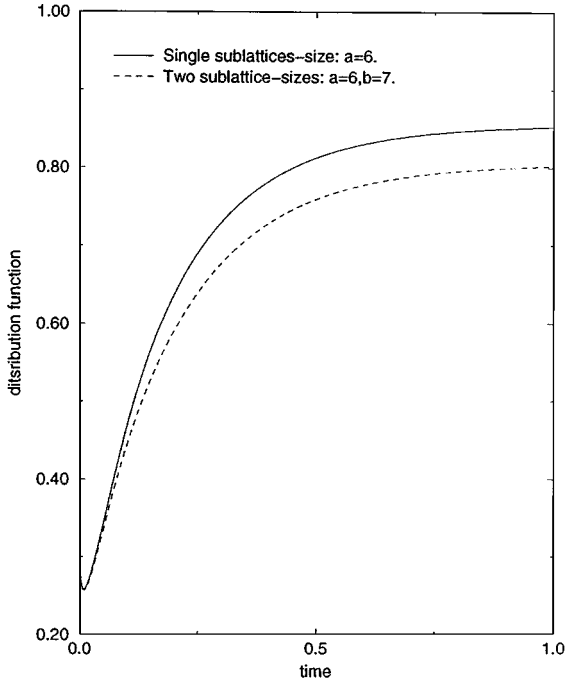

FIG. 11. Relaxation of $f(v_0)$ using one and two sublattice sizes.

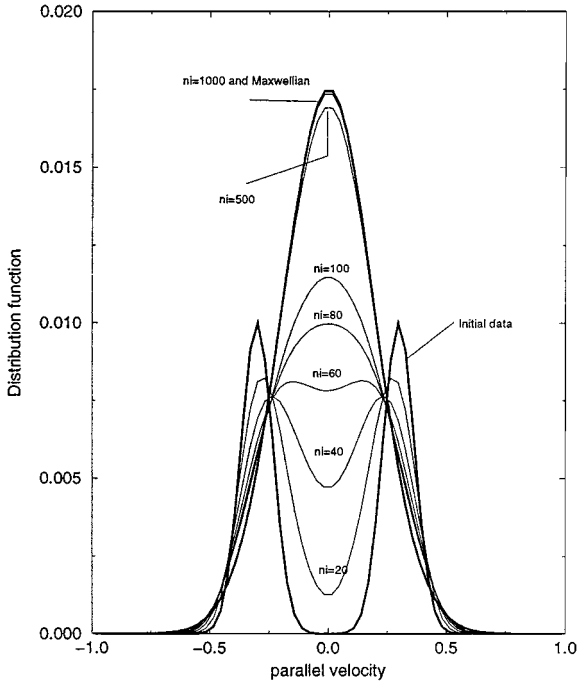

FIG. 12. Test with the initial data used in [18, 11].

TABLE III

N	17×8	33×16			65×32	
Sublattices sizes	2, 3	2, 3	3, 4	5, 6	7, 8	10, 11
Sublattices	0.005 s	0.06 s	0.03 s	0.015 s	0.16 s	0.07 s
Multigrids MC	0.02 s	0.07 s			0.45 s	
Multipole	0.01 s	0.005 s			0.3 s	
Quadratic schemes	0.018 s	0.3 s			6.3 s	

with the quadratic scheme. Finally, note that the Coulombian computations are a little bit more expensive than Maxwellian ones because of the presence of a non-zero power of the relative velocity in Coulombian interactions ($\gamma = -3$). Table III shows the CPU times (per iteration in time) for the Coulombian case.

6. CONCLUSION

We have established a simplified expression of the FPL operator in a cylindrical geometry. As for the three-dimensional case [6], this expression is written in a weak formulation from which we have derived conservative and entropic discretizations. A symmetrization and appropriate discretizations have been used to treat the problem near the axis (for the cylindrical geometry). A second part of this work was concerned with the application of fast algorithms to such discretizations. These algorithms were already shown to be efficient in the three-dimensional case [5]. Various numerical tests were presented in this paper and comparisons between these fast methods in both Coulombian and Maxwellian cases were given.

The axisymmetric FPL eqation is of great interest for physical applications mainly in the laser–plasma interactions and in astrophysics areas. For that purpose, the study of the space inhomogeneouse case is necessary and will be investigated in a future work.

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